Language: **English**

Day: **1**

April 2020

Problem 1. The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \quad \text{for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

Problem 2. Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

(i) $x_1 \leq x_2 \leq \ldots \leq x_{2020}$;

(ii) $x_{2020} \leq x_1 + 1$;

(iii) there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that

$$\sum_{i=1}^{2020} \left((x_i + 1)(y_i + 1)\right)^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ *is a permutation of* $(1, 2, 2)$*, and they are both permutations of* $(2, 2, 1)$*. Note that any list is a permutation of itself.*

Problem 3. Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent.

Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.

Note that $\angle A = \angle FAB$*. The other interior angles of the hexagon are similarly described.*

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

To make this a fair and enjoyable contest for everyone, please do not mention or refer to the problems on the internet or on social media until Saturday 18 April, 22:00 UTC (15:00 Pacific Daylight Time, 23:00 British Summer Time, 08:00 (Sunday) Australian Eastern Standard Time).

Language: **English**

Day: **2**

April 2020

Problem 4. A permutation of the integers 1, 2, $\ldots$, m is called *fresh* if there exists no positive integer $k < m$ such that the first k numbers in the permutation are 1, 2, $\ldots$, k in some order. Let f_m be the number of fresh permutations of the integers 1, 2, $\ldots$, m.

Prove that $f_n \geq n \cdot f_{n-1}$ for all $n \geq 3$.

For example, if $m = 4$, then the permutation $(3, 1, 4, 2)$ is fresh, whereas the permutation $(2, 3, 1, 4)$ is not.

Problem 5. Consider the triangle ABC with $\angle BCA > 90^\circ$. The circumcircle Γ of ABC has radius R. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA is R. The perpendicular bisector of PB intersects Γ at the points D and E.

Prove that P is the incentre of triangle CDE.

Problem 6. Let $m > 1$ be an integer. A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = a_2 = 1$, $a_3 = 4$, and for all $n \geq 4$,

$$a_n = m(a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all integers m such that every term of the sequence is a square.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

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