61th International Mathematical Olympiad

Day 2. Official Solutions

Problem 4. There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are *linked* by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Answer: $k = n^2 - n + 1$.

Solution. Number the stations by $1, 2, \dots, n^2$ from the bottom to the top.

We start with showing that for any $k \leqslant n^2 - n$ there may be no pair of stations linked by both companies. Clearly, it suffices to provide such an example for $k = n^2 - n$.

Let company A connect the pairs of stations of the form $(i, i+1)$, where $n \nmid i$. Then all pairs of stations (i, j) linked by A satisfy $\lceil i/n \rceil = \lceil j/n \rceil$.

Let company B connect the pairs of the form $(i, i+n)$, where $1 \leqslant i \leqslant n^2 - n$. Then pairs of stations (i, j) linked by B satisfy $i \equiv j \pmod{n}$. Clearly, no pair (i, j) satisfies both conditions, so there is no pair linked by both companies.

Now we show that for $k = n^2 - n + 1$ there always exist two required stations. Define an *A-chain* as a sequence of stations $a_1 < a_2 < \ldots < a_t$ such that company A connects a_i with a_{i+1} for all $1 \leqslant i \leqslant t-1$, but there is no A-car transferring from some station to a_1 and no A-car transferring from a_t to any other station. Define *B-chains* similarly. Moving forth and back, one easily sees that any station is included in a unique A-chain (possibly consisting of that single station), as well as in a unique B-chain. Now, put each station into a correspondence to the pair of the A-chain and the B-chain it belongs to.

All finishing points of A-cars are distinct, so there are $n^2 - k = n - 1$ stations that are not such finishing points. Each of them is a starting point of a unique A-chain, so the number of A-chains is $n-1$. Similarly, the number of B-chains also equals $n-1$. Hence, there are $(n-1)^2$ pairs consisting of an A- and a B-chain. Therefore, two of the n^2 stations correspond to the same pair, so that they belong to the same A-chain, as well as to the same B-chain. This means that they are linked by both companies, as required.

Comment 1. The condition that a car which starts higher also finishes higher is not used in the above solution.

Comment 2. If the number of stations were N, then the answer would be $N - \lceil \sqrt{N} \rceil + 1$. The solution above works verbatim for this generalization.

Problem 5. A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards.

For which n does it follow that the numbers on the cards are all equal?

Answer: For all integer $n > 1$.

Solution 1. Suppose that the numbers $a_1, \ldots, a_n$ written on the cards are not all equal. Let $d = \gcd(a_1 \ldots, a_n)$. If $d > 1$ then replace the numbers $a_1, \ldots, a_n$ by $\frac{a_1}{d}, \ldots, \frac{a_n}{d}$; all arithmetic and all geometric means will be divided by d, so we obtain another deck of cards satisfying the condition. Hence, without loss of generality, we can assume that $\gcd(a_1 \ldots, a_n) = 1$.

We show two numbers, a_m and a_k such that their arithmetic mean, $\frac{a_m + a_k}{2}$ is different from the geometric mean of any (nonempty) subsequence of $a_1 \ldots, a_n$, thus reaching a contradiction.

Choose the index $m \in \{1, \ldots, n\}$ such that $a_m = \max(a_1, \ldots, a_n)$. Note that $a_m \geqslant 2$, because $a_1, \ldots, a_n$ are not all equal. Let p be a prime divisor of a_m.

Let $k \in \{1, \ldots, n\}$ be an index such that $a_k = \max\{a_i : p \nmid a_i\}$. Due to $\gcd(a_1 \ldots, a_n) = 1$, not all a_i are divisible by p, so such a k exists. Note that $a_m > a_k$ because $a_m \geqslant a_k$, $p \mid a_m$ and $p \nmid a_k$.

Let $b = \frac{a_m + a_k}{2}$; we will show that b cannot be the geometric mean of any subsequence of $a_1, \ldots, a_n$.

Consider the geometric mean, $g = \sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}}$ of an arbitrary subsequence of $a_1, \ldots, a_n$.

If none of $a_{i_1}, \ldots, a_{i_t}$ is divisible by p, then they are not greater than a_k, so

$$g = \sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}} \leqslant a_k < \frac{a_m + a_k}{2} = b,$$

and therefore $g \neq b$.

Otherwise, if at least one of $a_{i_1}, \ldots, a_{i_t}$ is divisible by p, then $2g = 2\sqrt[t]{a_{i_1} \cdot \ldots \cdot a_{i_t}}$ is either not an integer or is divisible by p, while $2b = a_m + a_k$ is an integer not divisible by p, so $g \neq b$ again.

Solution 2. Like in the previous solution, we argue indirectly and assume that the numbers $a_1, \ldots, a_n$ written on the cards are not all equal and have no common divisor greater than 1. The arithmetic mean of any two numbers on two cards is half of an integer; on the other hand, it is a (some integer order) root of an integer. This means each pair's mean is an integer, so all numbers on the cards must be of the same parity; hence they all are odd. Let $d = \min\left\{ \gcd(a_i, a_j) \colon a_i \neq a_j \right\}$. By renumbering the cards we can assume that $\gcd(a_1, a_2) = d$, the sum $a_1 + a_2$ is maximal among such pairs, and $a_1 > a_2$.

We will show that $\frac{a_1 + a_2}{2}$ cannot be the geometric mean of any subsequence of $a_1 \ldots, a_n$.

Let $a_1 = xd$ and $a_2 = yd$ where x, y are coprime, and suppose that there exist some $b_1, \ldots, b_t \in \{a_1, \ldots, a_n\}$ whose geometric mean is $\frac{a_1 + a_2}{2}$. Let $d_i = \gcd(a_1, b_i)$ for $i = 1, 2, \ldots, t$ and let $D = d_1 d_2 \cdot \ldots \cdot d_t$. Then

$$D = d_1 d_2 \cdot \ldots \cdot d_t \mid b_1 b_2 \cdot \ldots \cdot b_t = \left(\frac{a_1 + a_2}{2}\right)^t = \left(\frac{x + y}{2}\right)^t d^t.$$

We claim that $D \mid d^t$. Consider an arbitrary prime divisor p of D. Let $\nu_p(x)$ denote the exponent of p in the prime factorization of x. If $p \mid \frac{x+y}{2}$, then $p \nmid x, y$, so p is coprime with x; hence, $\nu_p(d_i) \leqslant \nu_p(a_1) = \nu_p(xd) = \nu_p(d)$ for every $1 \leqslant i \leqslant t$, therefore $\nu_p(D) = \sum_i \nu_p(d_i) \leqslant t\nu_p(d) = \nu_p(d^t)$. Otherwise, if p is coprime to $\frac{x+y}{2}$, we have $\nu_p(D) \leqslant \nu_p(d^t)$ trivially. The claim has been proved.

Notice that $d_i = \gcd(b_i, a_1) \geqslant d$ for $1 \leqslant i \leqslant t$: if $b_i \neq a_1$ then this follows from the definition of d; otherwise we have $b_i = a_1$, so $d_i = a_1 \geqslant d$. Hence, $D = d_1 \cdot \ldots \cdot d_t \geqslant d^t$, and the claim forces $d_1 = \ldots = d_t = d$.

Finally, by $\frac{a_1+a_2}{2} > a_2$ there must be some b_k which is greater than a_2. From $a_1 > a_2 \geqslant d = \gcd(a_1, b_k)$ it follows that $a_1 \neq b_k$. Now the have a pair a_1, b_k such that $\gcd(a_1, b_k) = d$ but $a_1 + b_k > a_1 + a_2$; that contradicts the choice of a_1 and a_2.

Problem 6. Prove that there exists a positive constant c such that the following statement is true:

Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ *separates* a set of points $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

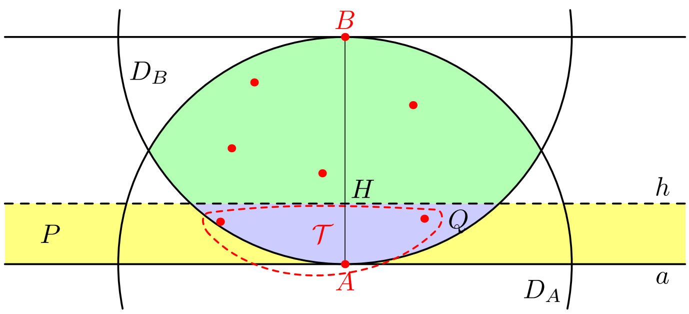
Note. Weaker results with $cn^{-1/3}$ replaced by $cn^{-\alpha}$ may be awarded points depending on the value of the constant $\alpha > 1/3$.

Solution. We prove that the desired statement is true with $c = \frac{1}{8}$. Set $\delta = \frac{1}{8}n^{-1/3}$. For any line ℓ and any point X, let X_ℓ denote the projection of X to ℓ; a similar notation applies to sets of points.

Suppose that, for some line ℓ, the set $\mathcal{S}_\ell$ contains two adjacent points X and Y with $XY = 2d$. Then the line perpendicular to ℓ and passing through the midpoint of segment XY separates $\mathcal{S}$, and all points in $\mathcal{S}$ are at least d apart from ℓ. Thus, if $d \geqslant \delta$, then a desired line has been found. For the sake of contradiction, we assume that no such points exist, in any projection.

Choose two points A and B in $\mathcal{S}$ with the maximal distance $M = AB$ (i.e., AB is a *diameter* of $\mathcal{S}$); by the problem condition, $M \geqslant 1$. Denote by ℓ the line AB. The set $\mathcal{S}$ is contained in the intersection of two disks D_A and D_B of radius M centered at A and B, respectively. Hence, the projection $\mathcal{S}_\ell$ is contained in the segment AB. Moreover, the points in $\mathcal{S}_\ell$ divide that segment into at most $n-1$ parts, each of length less than 2δ. Therefore,

$$M < n \cdot 2\delta. \tag{1}$$



Choose a point H on segment AB with $AH = \frac{1}{2}$. Let P be a strip between the lines a and h perpendicular to AB and passing through A and H, respectively; we assume that P contains its boundary, which consists of lines a and h. Set $\mathcal{T} = P \cap \mathcal{S}$ and let $t = |\mathcal{T}|$. By our assumption, segment AH contains at least $\left\lceil \frac{1}{2} : (2\delta) \right\rceil$ points of S_ℓ, which yields

$$t \geqslant \frac{1}{4\delta}. \tag{2}$$

Notice that $\mathcal{T}$ is contained in $Q = P \cap D_B$. The set Q is a circular segment, and its projection Q_a is a line segment of length

$$2\sqrt{M^2 - \left(M - \frac{1}{2}\right)^2} < 2\sqrt{M}.$$

On the other hand, for any two points $X, Y \in \mathcal{T}$, we have $XY \geqslant 1$ and $X_\ell Y_\ell \leqslant \frac{1}{2}$, so $X_a Y_a = \sqrt{XY^2 - X_\ell Y_\ell^2} \geqslant \frac{\sqrt{3}}{2}$. To summarize, t points constituting $\mathcal{T}_a$ lie on the segment of length less than $2\sqrt{M}$, and are at least $\frac{\sqrt{3}}{2}$ apart from each other. This yields $2\sqrt{M} > (t-1)\frac{\sqrt{3}}{2}$, or

$$t < 1 + \frac{4\sqrt{M}}{\sqrt{3}} < 4\sqrt{M}, \tag{3}$$

as $M \geqslant 1$.

Combining the estimates (1), (2), and (3), we finally obtain

$$\frac{1}{4\delta} \leqslant t < 4\sqrt{M} < 4\sqrt{2n\delta}, \quad \text{or} \quad 512n\delta^3 > 1,$$

which does not hold for the chosen value of δ.

Comment 1. As the proposer mentions, the exponent $-1/3$ in the problem statement is optimal. In fact, for any $n \geqslant 2$, there is a configuration $\mathcal{S}$ of n points in the plane such that any two points in $\mathcal{S}$ are at least 1 apart, but every line ℓ separating $\mathcal{S}$ is at most $c'n^{-1/3}\log n$ apart from some point in $\mathcal{S}$; here c' is some absolute constant.

On the other hand, it is much easier to prove the estimate of the form $cn^{-1/2}$. E.g., setting $\delta = \frac{1}{16}n^{-1/2}$ and applying (1), we see that $\mathcal{S}$ is contained in a disk D of radius $\frac{1}{8}n^{1/2}$. On the other hand, for each point X of $\mathcal{S}$, let D_X be the disk of radius $\frac{1}{2}$ centered at X; all these disks have disjoint interiors and lie within the disk concentric to D, of radius $\frac{1}{16}n^{1/2} + \frac{1}{2} < \frac{1}{2}n^{1/2}$. Comparing the areas, we get

$$n \cdot \frac{\pi}{4} \leqslant \pi\left(\frac{n^{1/2}}{16} + \frac{1}{2}\right)^2 < \frac{\pi n}{4},$$

which is a contradiction.

Comment 2. In this comment, we discuss some versions of the solution above, which avoid concentrating on the diameter of $\mathcal{S}$. We start with introducing some terminology suitable for those versions.

Put $\delta = cn^{-1/3}$ for a certain sufficiently small positive constant c. For the sake of contradiction, suppose that, for some set $\mathcal{S}$ satisfying the conditions in the problem statement, there is no separating line which is at least δ apart from each point of $\mathcal{S}$.

Let C be the convex hull of $\mathcal{S}$. A line is separating if and only if it meets C (we assume that a line passing through a point of $\mathcal{S}$ is always separating). Consider a strip between two parallel separating lines a and a' which are, say, $\frac{1}{4}$ apart from each other. Define a *slice* determined by the strip as the intersection of $\mathcal{S}$ with the strip. The *length* of the slice is the diameter of the projection of the slice to a.

In this terminology, the arguments used in the proofs of (2) and (3) show that for any slice $\mathcal{T}$ of length L, we have

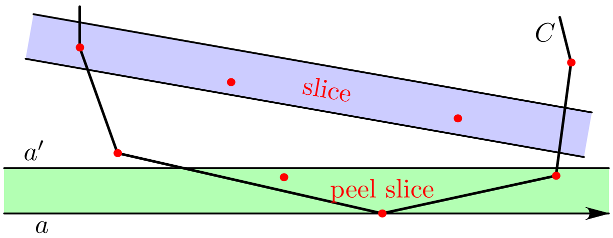
$$\frac{1}{8\delta} \leqslant |\mathcal{T}| \leqslant 1 + \frac{4}{\sqrt{15}}L. \tag{4}$$

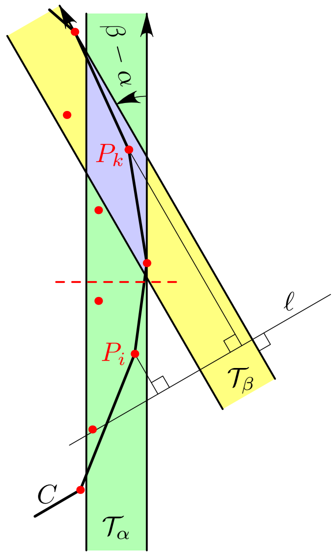
The key idea of the solution is to apply these estimates to a *peel slice*, where line a does not cross the interior of C. In the above solution, this idea was applied to one carefully chosen peel slice. Here, we outline some different approach involving many of them. We always assume that n is sufficiently large.

Consider a peel slice determined by lines a and a', where a contains no interior points of C. We orient a so that C lies to the left of a. Line a is called a *supporting line* of the slice, and the obtained direction is the *direction* of the slice; notice that the direction determines uniquely the supporting line and hence the slice. Fix some direction $\mathbf{v}_0$, and for each $\alpha \in [0, 2\pi)$ denote by $\mathcal{T}_\alpha$ the peel slice whose direction is $\mathbf{v}_0$ rotated by α counterclockwise.

When speaking about the slice, we always assume that the figure is rotated so that its direction is vertical from the bottom to the top; then the points in $\mathcal{T}$ get a natural order from the bottom to the top. In particular, we may speak about the *top half* $\mathsf{T}(\mathcal{T})$ consisting of $\lfloor |\mathcal{T}|/2 \rfloor$ topmost points in $\mathcal{T}$, and similarly about its *bottom half* $\mathsf{B}(\mathcal{T})$. By (4), each half contains at least 10 points when n is large.

Claim. Consider two angles $\alpha, \beta \in [0, \pi/2]$ with $\beta - \alpha \geqslant 40\delta =: \phi$. Then all common points of $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ lie in $\mathsf{T}(\mathcal{T}_\alpha) \cap \mathsf{B}(\mathcal{T}_\beta)$.





Proof. By symmetry, it suffices to show that all those points lie in $\mathsf{T}(\mathcal{T}_\alpha)$. Let a be the supporting line of $\mathcal{T}_\alpha$, and let ℓ be a line perpendicular to the direction of $\mathcal{T}_\beta$. Let $P_1, \dots, P_k$ list all points in $\mathcal{T}_\alpha$, numbered from the bottom to the top; by (4), we have $k \geqslant \frac{1}{8}\delta^{-1}$.

Introduce the Cartesian coordinates so that the (oriented) line a is the y-axis. Let P_i be any point in $\mathsf{B}(\mathcal{T}_\alpha)$. The difference of ordinates of P_k and P_i is at least $\frac{\sqrt{15}}{4}(k-i) > \frac{1}{3}k$, while their abscissas differ by at most $\frac{1}{4}$. This easily yields that the projections of those points to ℓ are at least

$$\frac{k}{3}\sin\phi - \frac{1}{4} \geqslant \frac{1}{24\delta} \cdot 20\delta - \frac{1}{4} > \frac{1}{4}$$

apart from each other, and P_k is closer to the supporting line of $\mathcal{T}_\beta$ than P_i, so that P_i does not belong to $\mathcal{T}_\beta$. □

Now, put $\alpha_i = 40\delta i$, for $i = 0, 1, \dots, \left\lfloor \frac{1}{40}\delta^{-1} \cdot \frac{\pi}{2} \right\rfloor$, and consider the slices $\mathcal{T}_{\alpha_i}$. The Claim yields that each point in $\mathcal{S}$ is contained in at most two such slices. Hence, the union $\mathcal{U}$ of those slices contains at least

$$\frac{1}{2} \cdot \frac{1}{8\delta} \cdot \frac{1}{40\delta} \cdot \frac{\pi}{2} = \frac{\lambda}{\delta^2}$$

points (for some constant λ), and each point in $\mathcal{U}$ is at most $\frac{1}{4}$ apart from the boundary of C.

It is not hard now to reach a contradiction with (1). E.g., for each point $X \in \mathcal{U}$, consider a closest point $f(X)$ on the boundary of C. Obviously, $f(X)f(Y) \geqslant XY - \frac{1}{2} \geqslant \frac{1}{2}XY$ for all $X, Y \in \mathcal{U}$. This yields that the perimeter of C is at least $\mu\delta^{-2}$, for some constant μ, and hence the diameter of $\mathcal{S}$ is of the same order.

Alternatively, one may show that the projection of $\mathcal{U}$ to the line at the angle of $\pi/4$ with $\mathbf{v}_0$ has diameter at least $\mu\delta^{-2}$ for some constant μ.